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THE FLOW FIELD INDUCED BY THE TORSIONAL OSCILLATIONS OF A SPHERICAL CELL CONTAINING A FLUID DROP

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Abstract—The flow field induced by the torsional oscillations of a spherical cell containing a fluid drop is examined. It has been found that in addition to the oscillating motion of each fluid particle there exist three standing vortices in every quadrant between the drop and the container. The steady streaming into the drop can be directed either clockwise or counterclockwise, depending on the values of the parameter of the fluids inside and outside the drop. Typical flow fields are shown graphically.

1. INTRODUCTION

The fact that an oscillating boundary can produce steady streaming in fluids has been known for some time. In a theoretical work, Schlichting (1932) found a periodic solution for the problem of an oscillating circular cylinder in an unbounded viscous fluid by the method of the two dimensional nonsteady boundary layer equations. It was found that only the boundary conditions at the cylinder can be satisfied, and that at a large distance the tangential component of velocity (V_θ) is finite but not zero. Unable to satisfy the condition $V_\theta = 0$ at $r = \infty$ he relaxed it to a stipulation that the tangential velocity remains bounded at the edge of the boundary layer. This result indicates that a potential motion which is periodic with respect to time induces a steady streaming at a large distance from the wall of the body. The case of an oscillating circular cylinder has also been investigated theoretically by Andres & Ingard (1953) and Holtsmark *et al.* (1954). Following the same method of analysis, Lane (1955) studied the problem of an oscillating sphere in an unbounded viscous fluid.

The rate of mass transfer between particles and oscillating fluids is a topic of great interest, an understanding of which is essential for predicting the magnitude of the beneficial effects and also for the design of oscillatory process equipment. Experimental verification of enhancement in mass transfer rates of solid–liquid boundaries by microstreaming has been found by Nybord (1965).

The beneficial effect of pulsating flow on mass transfer in liquid–liquid extraction apparatus has been studied by Krasuk & Smith (1963). They developed an analogy between mass and momentum transfer for predicting mass transfer coefficients.

It has been shown (Jameson 1964) that in certain circumstances the mechanism of the mass transfer is similar to that of heat transfer from a hot body in a fluctuating stream, in that the bulk of the transfer takes place in a steady streaming flow. A critical review of the mass transfer between solid spheres and oscillating fluids was given recently by Toweel & Landau (1976).

There is also experimental evidence that sound waves and vibrations affect heat transfer. A review of the subject was given by Richardson (1967).

Because of the difficulties associated with the three-dimensional analysis, steady streaming around spheres received much less attention (Wang 1965, Riley 1966). An examination of the flow field induced by a viscous fluid drop immersed in another translatorily oscillating unbounded fluid was undertaken by Zapryanov & Stoyanova (1978a, 1978b).

The torsional oscillations of cylinders and spheres have also been discussed both theoretically and experimentally. Di Prima & Liron (1976) pointed out that the torsional oscillation of a sphere in an unbounded viscous flow induced a secondary flow in the planes containing the axis of rotation and calculated the effect of the flow on the torque acting on the sphere.

Determining the behaviour of the steady streaming in the case of a bounded flow is an interesting, yet unanswered question.

Tabakova & Zapryanov (1978) have investigated the unseady motion of the fluid between two concentric spheres when the inner one is forced to execute a torsional oscillation while the outer sphere remains at rest. Recently Duck & Smith (1979) have studied the flow field induced between oscillating cylinders. Munson & Douglas (1979) have presented theoretical and experimental results describing viscous incompressible flow in spherical annuli. Their theoretical results are valid for low frequency oscillations.

The objective of the present work is to investigate the behaviour of the flow field induced by the torsional oscillations of a spherical cell containing a fluid drop. It is admitted that the fluids inside and outside of the drop are immiscible. It is assumed that the drop is a liquid sphere whose center coincides with the center of the container. Therefore we neglect the effects of shape-mode resonances and their associated violent mechanical deformation at the globular surface. Of particular interest is the steady streaming induced both inside and outside the drop.

The fluid motion is governed by a pair of coupled nonlinear partial differential equations in two independent variables, with singular end conditions. The problem has been solved in the case of high frequency oscillations of the container by the method of matched asymptotic expansions.

If a body of typical dimension a oscillates with the velocity $U_0 \cdot \cos \omega_0 t$ in a fluid of kinematic viscosity ν, which is otherwise at rest we can display the following dimensionless parameters which have appeared

$$\epsilon = \frac{U_0}{\omega_0 a}, \quad M^2 = \frac{a^2\omega_0}{\nu}, \quad \mathrm{Re} = \frac{U_0 a}{\nu}, \quad \mathrm{Re}\, s = \frac{U_0^2}{\omega_0 \nu}. \qquad [1]$$

Since $\mathrm{Re} = \epsilon M^2$ and $\mathrm{Re}\, s = (\mathrm{Re}^2/M^2)$ only two are independent parameters.

Furthermore, we are concerned with the situation $\epsilon \ll 1$. Physically this condition implies that the amplitude of the oscillation is small compared with the radius of the drop a.

2. THEORY AND RESULTS

Consider a spherical fluid drop with radius a which is contained in a spherical cell with radius $b(a < b)$. Suppose that the two spheres are concentric and adopt spherical polar coordinates r', θ, α. Let the container execute a torsional oscillation with frequency ω_0 and angular amplitude ϵ, so that the angular velocity of the container is $\Omega = \epsilon\omega_0 e^{i\omega_0 t'}$. Here and in what follows only the real of any complex quantity will be considered.

We shall suppose that the amplitude of the oscillation is small and seek a solution that is independent of α. Dimensionless variables will be employed throughout the analysis, and physical parameters pertaining to the interior of the drop will be distinguished from the corresponding exterior parameters by a caret. The velocity components (V'_r, V'_θ) are related to the stream function Ψ' by

$$V'_r = \frac{1}{r'^2 \sin\theta}\frac{\partial \Psi'}{\partial \theta}, \quad V'_\theta = -\frac{1}{r' \sin\theta}\frac{\partial \Psi'}{\partial r'}. \qquad [2]$$

Dimensionless variables and parameters are now introduced according to the scheme:

$$r' = ar, \quad t' = \omega_0^{-1}\tau, \quad \Psi' = \epsilon a^3 \omega_0 \Psi, \quad V'_\alpha = \omega_0 a V_\alpha, \quad M^2 = \frac{\omega_0 a^2}{\nu}. \qquad [3]$$

The governing equations of unsteady-state motion for an incompressible Newtonian fluid in terms of non-dimensional variables are

$$\frac{\partial(D^2\Psi)}{\partial\tau}+\epsilon\left[\frac{2\Omega}{r^2\sin^2\theta}\left(\frac{\partial\Omega}{\partial r}\cos\theta-\frac{1}{r}\frac{\partial\Omega}{\partial\theta}\sin\theta\right)-\frac{1}{r^2\sin\theta}\frac{\partial(\Psi,D^2\Psi)}{\partial(r,\theta)}\right.$$
$$\left.+\frac{2D^2\Psi}{r^2\sin^2\theta}\left(\frac{\partial\Psi}{\partial r}\cos\theta-\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\sin\theta\right)\right]=\frac{1}{M^2}D^4\Psi \qquad [4]$$

$$\frac{\partial\Omega}{\partial\tau}-\frac{\epsilon}{r^2\sin\theta}\left[\frac{\partial\Psi}{\partial r}\frac{\partial\Omega}{\partial\theta}-\frac{\partial\Psi}{\partial\theta}\frac{\partial\Omega}{\partial r}\right]=\frac{1}{M^2}D^2\Omega. \qquad [5]$$

Here

$$D^2\Psi=\frac{\partial^2\Psi}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right),\quad \Omega=r\sin\theta V_\alpha$$

and V_α is the rotational speed. It is easy to find that $\hat{M}^2=(\gamma/\kappa)M^2$, where M, κ and γ are respectively the frequency parameter, the ratio of the viscosity of the interior to that of the exterior fluid and the ratio of the density of the interior to that of the exterior fluid.

The boundary conditions are

(1) No slip velocity on the wall of the cell

$$\Psi=\frac{\partial\psi}{\partial r}=0,\quad \Omega=r^2\sin^2\theta\, e^{i\tau}\quad\text{at}\quad r=\lambda=\frac{b}{a}. \qquad [6]$$

(2) Zero values of the normal velocity both inside and outside of the globule

$$\Psi=\hat{\Psi}=0\quad\text{at}\quad r=1. \qquad [7a]$$

(3) Continuity of tangential velocities across the interface, whereupon

$$\frac{\partial\Psi}{\partial r}=\frac{\partial\hat{\Psi}}{\partial r},\quad \Omega=\hat{\Omega}\quad\text{at}\quad r=1. \qquad [7b]$$

(4) Continuity of shear stresses of the two fluids across the interface

$$\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial\Psi}{\partial r}\right)=\kappa\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial\hat{\Psi}}{\partial r}\right)\quad\text{at}\quad r=1 \qquad [8]$$

$$\left(\frac{\partial\Omega}{\partial r}-\frac{2\Omega}{r}\right)=\kappa\left(\frac{\partial\hat{\Omega}}{\partial r}-\frac{2\hat{\Omega}}{r}\right)\quad\text{at}\quad r=1. \qquad [9]$$

These conditions hold when fluids are immiscible, the surface tension is constant, the surface viscosity effects are negligible and axial symmetry is postulated. It should be remarked at this point that one can use the condition requiring the balance of normal stresses to detetermine the shape of the fluid drop as function of Weber number $W_e=(\hat{\nu}\hat{\mu}'/a\sigma)$, κ, γ and M, where σ is the surface tension and $\hat{\mu}'$ the dynamic viscosity. Uniqueness of the solution is ensured by the additional conditions that the velocity be a finite continuous function of the position at the center $r=0$.

Reference to [4] and [5] show that in addition to the primary flow in the torsional direction there will be a secondary flow in the planes containing the axis of oscillation. Mathematical complexity of the problems [4]–[9] does not permit an analysis for arbitrary frequency. The parameter $1/M$ measures the thickness of the Stokes' layer $0(\sqrt{(\nu/\omega_0)})$ relative to the radius of the fluid drop. When $M \gg 1$ the space between the drop and the container is divided into three separate but overlapping regions: (i) two Stokes' layers of thickness $0[(\omega_0/2\nu)^{1/2}]$ adjacent to the fluid drop and the container, and (ii) an intermediate region between the two boundary layers.

We introduce the following variables:

(i) In the boundary layer around the drop

$$\eta = \frac{(r-1)M}{\sqrt{2}}, \quad \varphi = \frac{\Psi M}{\sqrt{2}}, \quad \omega. \tag{10}$$

(ii) In the boundary layer on the container

$$\zeta = \frac{(\lambda - r)M}{\sqrt{2}}, \quad \bar{\varphi} = \frac{\Psi M}{\sqrt{2}}, \quad \bar{\omega}. \tag{11}$$

(iii) In the boundary layer inside of the surface of the drop

$$\hat{\eta} = \frac{(1-r)M}{\sqrt{2}}, \quad \hat{\varphi} = \frac{\hat{\Psi} M}{\sqrt{2}}, \quad \hat{\omega}. \tag{12}$$

Here ω, $\bar{\omega}$ and $\hat{\omega}$ are the torsional velocities in the boundary layers around the drop, on the container and inside of the surface of the drop.

For small ϵ and large M the boundary value problem given by [4]–[9] can be solved by expanding in ϵ and $1/M$. We suppose that

$$F = F_{00} + \frac{1}{M} F_{01} + \cdots + \epsilon^2 \left(F_{10} + \frac{1}{M} F_{11} + \cdots \right) + \cdots, \tag{13}$$

where

$$F = [\omega, \Omega, \bar{\omega}, \hat{\omega}]$$

and

$$G = \epsilon \left[G_{00} + \frac{1}{M} G_{01} + \cdots \right] + \epsilon^3 \left[G_{10} + \frac{1}{M} G_{11} + \cdots \right] + \cdots \tag{14}$$

where

$$G = [\varphi, \Psi, \hat{\Psi}, \bar{\varphi}, \hat{\varphi}].$$

We look for an approximate solution for the stream function and the torsional velocity, valid asymptotically as $\epsilon \to 0$ and $M \to \infty$. The expressions obtained by truncating the expansions [13] and [14] after a given number of terms satisfy approximately the exact equations of motion [4] and [5] and the boundary conditions [6]–[9], with residuals which are bounded uniformly in separate but overlapping regions to known order in ϵ and $1/M$. We make the unknown functions in the intermediate region and the unknown functions in the Stokes' boundary layers match in the intermediate district. In this way, F_{00}, F_{10}, G_{00}, G_{10}, etc. can be determined successively and any arbitrary constants are fixed either immediately or at most a few stages further on. This procedure is now fairly standard, and we will present only the result, without entering into the detailed arguments showing why the terms must have the values given here.

Substituting [11] and [13] into [5] and equating the terms of equal powers of the angular displacement ϵ and frequency parameter $1/M$ we obtain the equations:

$$2\frac{\partial \bar{\omega}_{00}}{\partial \tau} - \frac{\partial^2 \bar{\omega}_{00}}{\partial \zeta^2} = 0 \tag{15}$$

$$\frac{\partial \bar{\omega}_{01}}{\partial \tau} - \frac{\partial^2 \bar{\omega}_{01}}{\partial \zeta^2} = 0 \qquad [16]$$

$$2\frac{\partial \bar{w}_{02}}{\partial \tau} - \frac{\partial^2 \bar{\omega}_{02}}{\partial \zeta^2} = \frac{2}{\lambda^2}(1-\mu^2)\frac{\partial^2 \bar{\omega}_{00}}{\partial \mu^2}. \qquad [17]$$

Equating the terms of equal powers in ϵ from [5] and [13] one can obtain that the functions Ω_i ($i = 0, 1, 2, \ldots$) satisfy the following equation:

$$\frac{\partial \Omega_i}{\partial \tau} = \frac{1}{M}\left[\frac{\partial^2 \Omega_i}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2 \Omega_i}{\partial \mu^2}\right]. \qquad [18]$$

First, the zero approximation of the circumferential velocity is computed.

Using the method of matched asymptotic expansions we have found

$$\bar{\omega}_{00} = \lambda^2 E(1-\mu^2)\,\mathrm{e}^{i\tau}, \quad \bar{\omega}_{01} = 0, \quad \bar{w}_{02} = -(1-i)\zeta(1-\mu^2)E\,\mathrm{e}^{i\tau}$$

$$\Omega_{00} = \Omega_{01} = \Omega_{02} \equiv 0, \quad \omega_{00} = \omega_{01} = \omega_{02} = 0, \quad \hat{\omega}_{00} = \hat{\omega}_{01} = \hat{\omega}_{02} = 0, \qquad [19]$$

where

$$\mu = \cos\theta \quad \text{and} \quad E = \mathrm{e}^{-(1+i)\zeta}. \qquad [20]$$

These results are used in computing the first approximation to the function of the secondary flow satisfying the equations:

$$\frac{1}{2}\frac{\partial^4 \bar{\Phi}_{00}}{\partial \zeta^4} - \frac{\partial^3 \bar{\varphi}_{00}}{\partial \tau \partial \zeta^2} = -\frac{2\cos\theta}{\lambda^2 \sin^2\theta}\bar{\omega}_{00}\frac{\partial \bar{\omega}_{00}}{\partial \zeta} \qquad [21]$$

$$\frac{1}{2}\frac{\partial^4 \bar{\varphi}_{01}}{\partial \zeta^4} - \frac{\partial^3 \bar{\varphi}_{01}}{\partial \tau \partial \zeta^2} = -\frac{4\sqrt{(2)}\cos\theta}{\lambda^2 \sin\theta}\zeta\bar{\omega}_{00}\frac{\partial \bar{\omega}_{00}}{\partial \zeta} - \frac{2\sqrt{(2)}}{\lambda^3 \sin\theta}\bar{\omega}_{00}\frac{\partial \bar{\omega}_{00}}{\partial \theta} \qquad [22]$$

$$\frac{1}{2}\frac{\partial^4 \bar{\varphi}_{02}}{\partial \zeta^4} - \frac{\partial^3 \bar{\varphi}_{02}}{\partial \tau \partial \zeta^2} = -\frac{2(1-\mu^2)}{\lambda^2}\frac{\partial^4 \bar{\varphi}_{10}}{\partial \zeta^2 \partial \mu^2} + \frac{2(1-\mu^2)}{\lambda^2}\frac{\partial^3 \bar{\varphi}_{10}}{\partial \tau \partial \mu^2} - \frac{12}{\lambda^2}\zeta^2\bar{\omega}_{00}\frac{\cos\theta}{\sin^2\theta}\frac{\partial \bar{\omega}_{00}}{\partial \zeta} - \frac{2\cos\theta}{\lambda^2\sin^2\theta}\bar{\omega}_{00}$$
$$\times\frac{\partial \bar{\omega}_{02}}{\partial \zeta} - \frac{2\cos\theta}{\lambda^2 \sin^2\theta}\bar{\omega}_{02}\frac{\partial \bar{\omega}_{00}}{\partial \zeta} - \frac{4\zeta}{\lambda^3 \sin\theta}\bar{\omega}_{02}\frac{\partial \bar{\omega}_{00}}{\partial \theta} - \frac{8\zeta}{\lambda^2 \sin\theta}\bar{\omega}_{00}\frac{\partial \bar{\omega}_{00}}{\partial \theta}. \qquad [23]$$

After some algebra we have obtained:

$$\bar{\varphi}_{00} = \lambda^2\mu(1-\mu^2)\left\{\frac{1}{8}(\mathrm{e}^{-23}-1) + \frac{1}{4}\zeta + \frac{\cos 2\tau - \sin 2\tau}{16}(1-\sqrt{(2)})\right.$$
$$\left. - \frac{\mathrm{e}^{-2\zeta}[\cos 2(\tau-\zeta) - \sin 2(\tau-\zeta)]}{16} + \frac{\mathrm{e}^{-\sqrt{(2)}\zeta}[\cos(2\tau - \sqrt{(2)}\zeta) - \sin(2\tau - \sqrt{(2)}\zeta)]}{8\sqrt{(2)}}\right\}$$

$$\bar{\varphi}_{01}^{(s)} = \left\{\left(1 - \frac{1}{8\lambda}\right)(\mathrm{e}^{-2\zeta}-1) + \left(\frac{15}{8} - \frac{1}{4\lambda}\right)\zeta + \frac{1}{8}\zeta\,\mathrm{e}^{-2\zeta}\right.$$
$$- \frac{\zeta^2}{8\lambda(4\lambda^{10} - 25\lambda^7 + 42\lambda^5 - 25\lambda^3 + 4)}\left[-25\lambda^7 + 7\lambda^{10} + 21\lambda^5 - 3\right.$$
$$\left.\left. + \frac{(15\lambda^7 - 35\lambda^5 + 35\lambda^2 - 15)\lambda^3(-30\lambda^7 + 105\lambda^2 - 75)}{-5\kappa(4\lambda^{10} - 25\lambda^7 + 42\lambda^5 - 25\lambda^3 + 4) + 2(4\lambda^{10} + 100\lambda^7 - 147\lambda^5 + 24\lambda^3 + 26}\right]\right\}\lambda^2 2\sqrt{(2)}\mu(1-\mu^2)$$

$$\bar{\varphi}_{01}^{(u)} = \left\{-\left(\frac{1}{\lambda} - 3\right)\frac{1}{16}\cos 2\tau + \frac{1}{2\sqrt{2}}\left\langle\frac{(1-\sqrt{2})(3\lambda^5+2)}{\lambda(1-\lambda^5)} + \frac{1}{8}\left(\frac{2}{\lambda} - 5\right)\right]\right.$$
$$\times[\mathrm{e}^{-\sqrt{(2)}\zeta}\cos(2\tau - \sqrt{(2)}\zeta) + \sqrt{(2)}\zeta(\cos 2\tau - \sin 2\tau) - \cos 2\tau]$$
$$-\frac{\zeta}{16}\left[\left(\frac{2}{\lambda} - 5\right)(\cos 2\tau - \sin 2\tau) - \mathrm{e}^{-2\zeta}\cos 2(\tau-\zeta) - \sin 2(\tau-\zeta)\right]$$
$$\left. + \left(\frac{1}{\lambda} - 3\right)\frac{\cos 2(\tau-\zeta) - \sin 2(\tau-\zeta)}{16\,\mathrm{e}^{2\zeta}}\right\}2\sqrt{(2)}\mu(1-\mu^2)\lambda^2$$

where the superscript s denotes steady and u unsteady state. The equations for φ_{00}, φ_{01} and φ_{02} are like [21]–[23]. It is easy to find that

$$\varphi_{00} = -\frac{(15\lambda^7 - 35\lambda^5 + 35\lambda^2 - 15)\lambda^3\eta\mu(1-\mu^2)}{2[-5\kappa(4\lambda^{10} - 25\lambda^7 + 42\lambda^5 - 25\lambda^3 + 4) + 2(-4\lambda^{10} + 100\lambda^7 - 147\lambda^5 + 25\lambda^3 + 26]}$$

$$\varphi_{01}^{(s)} = \left\{-\frac{(450\lambda^8 - 75\lambda^7 - 510\lambda^6 + 205\lambda^5 + 1050\lambda^3 - 138\lambda^2 - 450\lambda + 30)\eta}{2\sqrt{(2)}[-5\kappa(4\lambda^{10} - 25\lambda^7 + 42\lambda^5 + 25\lambda^3 + 4) + 4\lambda^{10} + 50\lambda^7 - 24\lambda^5 + 10\lambda^3 - 10\lambda + 20]}\right.$$
$$\left. + \frac{(5\kappa + 2)(15\lambda^7 - 35\lambda^5 + 35\lambda^2 + 15)\eta^2}{2[-5\kappa(4\lambda^{10} - 25\lambda^7 + 42\lambda^5 - 25\lambda^3 + 4) + 2(-4\lambda^{10} + 100\lambda^7 - 147\lambda^5 + 25\lambda^3 + 26)]}\right\}$$
$$\times \lambda^2\mu(1-\mu^2)$$

$$\varphi_{01}^{(u)} = \left\{\frac{5\sqrt{(2)}(\sqrt{(2)} - 1)\lambda^2}{1 - \lambda^5}(\cos 2\tau - \sin 2\tau)\eta\right.$$
$$\left. + \frac{5(\sqrt{(2)} - 1)\lambda^2\kappa}{(4+\kappa)(1-\lambda^5)}[e^{-\sqrt{(2)}\eta}\cos(2\tau - \sqrt{(2)}\eta) - \cos 2\tau]\right\}\lambda^2\mu(1-\mu^2).$$

In the region between the boundary layers around the drop and on the container we will have

$$\frac{\partial}{\partial\tau}(D^2\Psi_0) = \frac{1}{M^2}D^4\Psi_0 \tag{[24]}$$

where

$$\Psi_0 = \Psi_{00} + \frac{1}{M}\Psi_{01} + \frac{1}{M}\Psi_{02} + \cdots.$$

In addition to the oscillatory term we expect Ψ_0 to contain an independent of τ term

$$\Psi_0 = \Psi_0^{(s)} + e^{2\tau i}\Psi_0^{(u)}. \tag{[25]}$$

The equation for the unsteady and steady parts of Ψ_0 are

$$D^2\Psi_{00}^{(u)} = 0, \quad D^2\Psi_{01}^{(u)} = 0, \quad D^2\Psi_{02}^{(u)} = -\frac{i}{2}D^4\Psi_{00}^{(u)} \tag{[26]}$$

$$D^4\Psi_{00}^{(s)} = 0, \quad D^4\Psi_{01}^{(5)} = 0, \quad D^4\Psi_{02}^{(s)} = 0. \tag{[27]}$$

The functions Ψ_{00} and Ψ_{01} are found to be

$$\Psi_{00} = \frac{1}{2}\cdot\frac{\lambda^3\mu(1-\mu^2)}{4\lambda^{10} - 25\lambda^7 + 42\lambda^5 - 25\lambda^3 + 4}\left\{\frac{1}{2}[(-2\lambda^5 + 25\lambda^2 - 3)r^5\right.$$
$$-(-2\lambda^7 + 7\lambda^2 - 5)r^3 + (-5\lambda^7 + 3\lambda^5 - 2) - (3\lambda^5 + 5\lambda^3 - 2)\lambda^2 r^{-2}]$$
$$\left. + \frac{(15\lambda^7 - 35\lambda^5 + 35\lambda^2 - 15)[6(1-\lambda^5)r^5 + 10(\lambda^7 - 1)r^3 - 25\lambda^7 + 21\lambda^5 + 4 + 15(\lambda^4 - \lambda^2)\lambda^2 r^{-2}]}{-5\kappa(4\lambda^{10} - 25\lambda^7 + 42\lambda^5 - 25\lambda^3 + 4) + 2(-4\lambda^{10} + 100\lambda^7 - 147\lambda^5 + 25\lambda^3 + 26)}\right\}$$

$$\Psi_{01}^{(u)} = \frac{\sqrt{(2)}(\sqrt{(2)} - 1)\lambda^4}{8(1-\lambda^5)}(\cos 2\tau - \sin 2\tau)(r^3 - r^{-2})\cdot\frac{1}{2}\mu(1-\mu^2)$$

$$\Psi_{01}^{(s)} = \left\{\left[(-30\lambda^6 + 7\lambda^5 + 75\lambda^3 - 10\lambda^2 - 45\lambda + 3)r^5 + (30\lambda^8 - 5\lambda^7 - 105\lambda^3 + 14\lambda^2 + 75\lambda - 5)r^3\right.\right.$$
$$+ \frac{-150\lambda^8 + 25\lambda^7 - 49\lambda^5 + 204\lambda^6 - 300\lambda + 50}{2}$$

$$+\frac{90\lambda^8-15\lambda^7-150\lambda^6+35\lambda^5+60\lambda^3-8\lambda^2}{2}r^2\Big]$$

$$+\frac{450\lambda^8-75\lambda^7-510\lambda^6+205\lambda^5+1050\lambda^3-138\lambda^2-450\lambda+30}{-5\kappa(4\lambda^{10}-25\lambda^7+42\lambda^5-25\lambda^3+4)+4\lambda^{10}+50\lambda^7+24\lambda^5+10\lambda^3-60\lambda+20}$$

$$\times[(3\lambda^5-2\lambda^3-3\lambda+2)r^5+(3\lambda^5-\lambda^7-2)r^3-(2\lambda^{10}+3\lambda^5-5\lambda^3)+(2\lambda^{10}+\lambda^7-3\lambda^5)r]\Big\}$$

$$\times\frac{\lambda^2\mu(1-\mu^2)}{\sqrt{(2)}(4\lambda^{10}-25\lambda^7+42\lambda^5-25\lambda^3+4)}.$$

Similarly, for the region inside the drop we have equations:

(i) In the boundary layer inside of the drop

$$L^2\frac{\partial^4}{\partial\hat{\eta}^4}\hat{\varphi}_{00}-2\frac{\partial^3\hat{\varphi}_{00}}{\partial\tau\partial\mu^2}=0$$

$$L^2\frac{\partial^4\hat{\varphi}_{01}}{\partial\hat{\eta}^4}-2\frac{\partial^3\hat{\varphi}_{01}}{\partial\tau\partial\mu^2}=0$$

$$L^2\frac{\partial^4\hat{\varphi}_{02}}{\partial\hat{\eta}^4}-2\frac{\partial^3\hat{\varphi}_{02}}{\partial\tau\partial\hat{\eta}^2}=-4(1-\mu^2)\frac{\partial^4\hat{\varphi}_{00}}{\partial\hat{\eta}^2\partial\mu^2}+4(1-\mu^2)\frac{\partial^3\hat{\varphi}_{00}}{\partial\tau\partial\mu^2}.$$

(ii) In the nucleus of the drop

$$D^2\hat{\Psi}_{00}^{(u)}=0,\quad D^2\hat{\Psi}_{01}^{(u)}=0,\quad D^2\hat{\Psi}_{02}^{(u)}=-iL^2D^y\hat{\Psi}_{00}^{(u)}$$

$$D^4\hat{\Psi}_{00}^{(s)}=0,\quad D^4\hat{\Psi}_{01}^{(s)}=0,\quad D^4\hat{\Psi}_{02}^{(s)}=0$$

where

$$L=\sqrt{\left(\frac{\kappa}{\gamma}\right)},\quad \kappa=\frac{\hat{\mu}'}{\mu'},\quad \gamma=\frac{\hat{\rho}}{\rho}.$$

For the functions in the boundary layer of the drop we have found

$$\hat{\varphi}_{00}=\frac{(15\lambda^7-35\lambda^5+35\lambda^2-15)\lambda^3\hat{\eta}\mu(1-\mu^2)}{2[-5\kappa(4\lambda^{10}-25\lambda^7+42\lambda^5-25\lambda^3+4)+2(-4\lambda^{10}+100\lambda^7-147\lambda^5+25\lambda^3+26)]}$$

$$\hat{\varphi}_{01}^{(u)}=\mu(1-\mu^2)\frac{5(\sqrt{(2)}-1)\lambda^4L^2}{(L+\kappa)(1-\lambda^5)}\left[e^{-(\sqrt{(2)}/L)\hat{\eta}}\cos\left(2\tau-\frac{\sqrt{2}}{L}\hat{\eta}\right)-\cos 2\tau\right]\lambda^2$$

$$\hat{\varphi}_{01}^{(s)}=\lambda^2\mu(1-\mu^2)$$

$$\Big\{\frac{(450\lambda^8-75\lambda^7-510\lambda^6+205\lambda^5+1050\lambda^3-138\lambda^2+450\lambda+30)\eta}{2\sqrt{(2)}[-5\kappa(4\lambda^{10}-25\lambda^7+42\lambda^5-25\lambda^3+4)+4\lambda^{10}+50\lambda^7-24\lambda^5+10\lambda^3-10\lambda+20]}$$

$$-\frac{7(15\lambda^7-35\lambda^5+35\lambda^2-15)\eta^2\lambda}{2\sqrt{2}[-5\kappa(4\lambda^{10}-25\lambda^7+42\lambda^5-25\lambda^3+4)+2(-4\lambda^{10}+100\lambda^7+147\lambda^5+25\lambda^3+26)]}\Big\}.$$

In the nucleus of the drop we have

$$\hat{\Psi}_{00}=-\frac{\mu(1-\mu^2)(15\lambda^7-35\lambda^5+35\lambda^2-15)\lambda^3(r^5-r^3)}{2[-5\kappa(4\lambda^{10}-25\lambda^7+42\lambda^5-25\lambda^3+4)+2(-4\lambda^{10}+100\lambda^7-147\lambda^5+25\lambda^3+26)]}$$

$$\hat{\Psi}_{01}^{(u)}=0$$

$$\hat{\Psi}_{01}^{(s)}=\frac{(450\lambda^8-75\lambda^5-510\lambda^6+205\lambda^5+1050\lambda^3-138\lambda^2-450\lambda+30)(r^5-r^3)\left(-\frac{1}{2}\right)\mu(1-\mu^2)\lambda^2}{2\sqrt{(2)}[-5\kappa(4\lambda^{10}-25\lambda^7+42\lambda^5-25\lambda^3+4)+4\lambda^{10}+50\lambda^7-24\lambda^5+10\lambda^3-60\lambda+20]}.$$

It is interesting to note that rotational speed is not equal to zero only in the Stokes shear layer on the container. In this way the primary flow is restricted to the boundary layer on the container. At the same time the secondary flow, i.e. the flow in the planes containing the axis of oscillation, has velocity components which are not equal to zero in the Stokes boundary layers and between them. There exists a nearly inviscid core in every quadrant between the boundary layers on the container and on the drop.

The character of the secondary flow depends strongly upon the dimensionless frequency and parameters of the fluids inside and outside the drop. Figure 1 shows the steady streaming of the secondary flow for a mercury drop in water ($M = 20$, $\lambda = 2$, $\kappa = 1$, $\gamma = 13.6$).

As can be seen from this figure, in addition to the vibrating motion of each fluid particle, we may have a pattern of three standing vortices, one in the inviscid core and two in the Stokes boundary layers. The circulation of the steady streaming in the core is directed clockwise but the two other vortices are directed counterclockwise. Figure 2 shows that when $M = 20$, $\lambda = 2$, $\gamma = 1.1$ and $\kappa = 0.001$ (for a water drop in Zerolene) there are two standing vortices in the secondary flow between the drop and the container. The steady streaming inside the drop is directed clockwise in figure 1 and counterclockwise in figure 2.

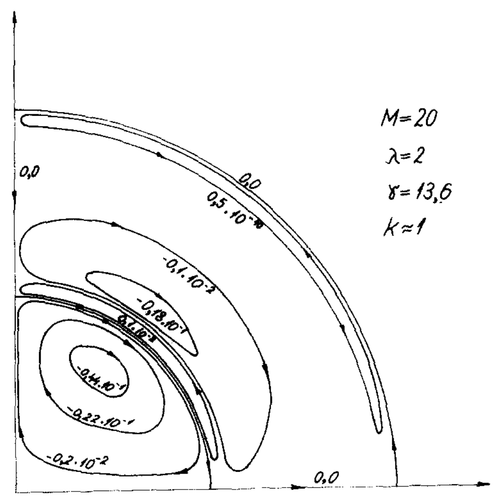

Figure 1. The stationary streamline pattern; $M = 20$, $\lambda = 2$, $\gamma = 13.6$, $\kappa \approx 1$.

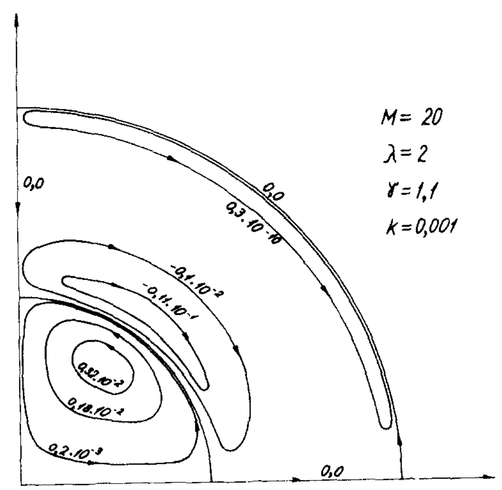

Figure 2. The stationary streamline pattern; $M = 20$, $\lambda = 2$, $\gamma = 1.1$, $\kappa = 0.001$.

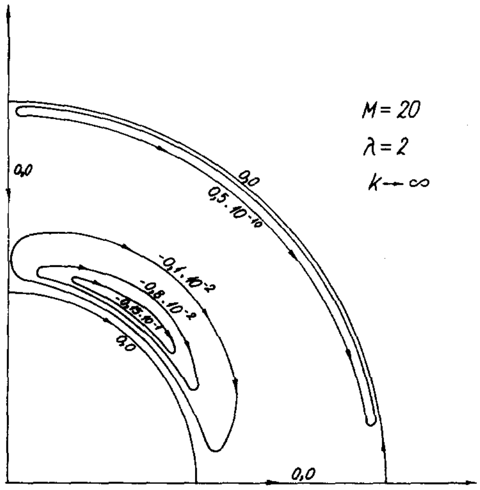

Figure 3. The stationary streamline pattern; $M = 20$, $\lambda = 2$, $\kappa \to \infty$.

It is interesting to note that for the limiting case of very viscous drops (rigid body, figure 3), the steady streaming in the secondary flow is similar to that of figure 2.

3. CONCLUSION

It has been found that, in the case of high frequency oscillations of the container, the primary flow is restricted to the boundary layer on the container. It becomes increasingly thinner as the frequency increases. Outside of this boundary layer the fluid motion is restricted almost entirely to the secondary flow in the planes containing the axis of rotation. There is a nearly inviscid core between the boundary layers on the container and on the drop. The flow of the core is driven by the boundary layer on the container.

The character of the flow in the planes containing the axis of rotation depends strongly upon the dimensionless frequency and parameters of the fluids inside and outside the drop. In addition to the vibrating motion of each fluid particle we may have a pattern of three standing vortices (time independent circulations) in every quadrant between the fluid drop and the container. Since the circulation of the steady stream in the core is directed clockwise the other two vortices are directed counterclockwise. This steady streaming results from the nonlinear coupling of the first order periodic fluid particle velocities and creates stirring action which helps to achieve the above mentioned beneficial effects.

It was found that the direction of the steady streaming into the inviscid core between the fluid drop and the oscillating container is always directed clockwise. When the ratio of the viscosity of the fluids inside and outside the drop is very small there is not a standing vortex in the boundary layer around the drop and the steady streaming into it is directed counterclockwise (figure 2). In the case when there is a standing vortex in the boundary layer around the drop the steady streaming into it is directed clockwise (figure 1). At the same time for very viscous drops (rigid body, figure 3) the steady streaming into the region between the drop and the container is similar to that of figure 2.

REFERENCES

Andres, A. & Ingard, U. 1953 Acoustic streaming at high Reynolds numbers. *D. Acoust. Soc. Am.* **25**, 928–932.

Di Prima R. C. & Liron, N. 1976 Effect of secondary flow on the decaying torsional oscillations of a sphere and a plan. *Phys. Fluids* **19**, 1450–1458.

Duck, P. W. & Smith, F. T. 1979 Steady streaming induced between oscillating cylinders. *J. Fluid Mech.* **91**, 93–110.

HOLTSMARK, J., JANSEN, I., SIKKELAND, T. & CRAVLEM, S. 1954 Boundary layer flow near a cylindrical obstacle in an oscillating, incompressible fluid. *J. Acoust. Soc. Am.* **26**, 26–39.

JAMESON, G. J. 1964 Mass (or heat) transfer from an oscillating cylinder. *Chem. Engng Sci.* **19**, 793–800.

KRASUK, J. H. & SMITH, J. U. 1963 Mass transfer in a pulsed column. *Chem. Engng Sci.* **18**, 591–598.

LANE, C. 1955 Acoustical streaming in the vicinity of a sphere. *J. Acoust. Soc. Am.* **27**, 1082–1086.

MUNSON, B. R. & DOUGLASS, R. W. 1979 Viscous flow in oscillatory spherical annuli. *Phys. Fluids* **22**, 205–208.

NYBORG, W. L. 1965 In *Physical Acoustics* (Edited by W. P. Mason), Vol. 11B, pp. 265–331. Academic Press, New York.

RICHARDSON, P. D. 1967 Effect of sound and vibrations on heat transfer. *Appl. Mech. Rev.* **20**, 201–217.

RILEY, N. 1966 On the sphere oscillating in a viscous fluid. *Q. J. Mech. Appl. Math.* **19**, 461–472.

SCHLICHTING, H. 1932 Berechnung ebener periodischer Greenzschichtstromungen. *Physikalische Zeit.* **33**, 327–335.

TOWEEL, A. M. & LANDAU, J. 1976 Mass transfer between solid spheres and oscillating fluids—A critical review. *Can. J. Chem. Engng* **54**, 532–539.

WANG, C. Y. 1965 The flow field induced by an oscillating sphere. *J. Sound Vib.* **2**, 257–269.

ZAPRYANOV, Z. & STOYANOVA, S. 1978a A study of the fluid flow induced by a viscous fluid drop immersed in another oscillating fluid. *Z.A.M.M.* **68**, 355–356.

ZAPRYANOV, Z. & STOYANOVA, S. 1978b The flow field induced by an oscillating fluid drop immersed in another fluid. *Int. J. Multiphase flow* **4**, 193–201.

TABAKOVA, S. & ZAPRYANOV, Z. 1978 The flow induced by oscillating movement of solid particle in a spherical container. *Compt. Rend. Acad. Sci.* **7**, 819–822.